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Simulation of freezing of water-saturated porous media in a rectangular cavity under multiple heat sources with different temperature using a combined transfinite interpolation and PDE methods

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Abstract

A numerical study is made of the freezing of water-saturated porous media in a rectangular cavity subjected to multiple heat sources with different temperature. Focus is placed on establishing a computationally efficient approach for solving multi-dimensional moving boundary problem in a two-dimensional structured grids. Preliminary grids are first generated by an algebraic method, based on a transfinite interpolation method, with subsequent refinement using a PDE mapping (parabolic grid generation) method. A preliminary case study indicates successful implementation of the numerical procedure. A two-dimensional freezing model is validated against available analytical solution and experimental results. © 2006 Elsevier Ltd. All rights reserved.

Keywords: Freezing; Porous media; Transfinite interpolation; Moving boundary

1. Introduction

Transient heat transfer problems involving melting or solidification are generally referred to as "phase change" or "moving boundary" problems. They are an important topics which spans a broad spectrum of scientific and engineering disciplines such as the freezing or thawing of soil, ice formation, crystal growth, aerodynamic ablation, casting of metal, food processing and numerous others. Generally, the solution of moving boundary problem with phase transition has been of special interest due to the inherent difficulties associated with the non-linearity of the interface conditions and the unknown locations of the arbitrary moving boundaries. The some up to date reviews of these problems are available (Charn-Jung & Kaviany, 1992; Chellaiah & Viskanta, 1988; Frivik & Comini, 1982; Hasan, Mujumdar, & Weber, 1991; Hashemi & Sliepcevich, 1973; Murray & Landis, 1959; Sparrow & Broadbent, 1983; Weaver & Viskanta, 1986).

In the past, a variety of conventional numerical techniques have been developed for solving these problems, including the enthalpy (Crowley, 1978; Shamsundar & Sparrow, 1976), apparent heat capacity (Bonacina, Comini, Fasano, & Primicerio, 1973), isotherm migration (Crank & Gupta, 1975), and coordinate transformation methods (Cheung, Chawla, & Pedersen, 1984; Hsu, Sparrow, & Patankar, 1981; Sparrow & Chuck, 1984; Sparrow, Ramadhyani, & Patankar, 1978; Rattanadecho, 2004a, b). These methods have been introduced by researchers mainly to overcome the difficulties in handling moving boundaries. Previous works on multi-dimensional moving boundary problems include Duda, Malone, Notter, and Vrentas (1975), Saitoh (1978), Gong and Mujumdar (1998), Cao, Huang, and Russell (1999), Khillarkar, Gong, and Mujumdar (2000), Chatterjee and Prasad (2000), Beckett, MacKenzie, and Robertson (2001), and Ratanadecho, Aoki, and Akahori (2002).

Conventionally numerical methods have been widely used due to easy to handle numerical algorithms for phase change problem. However, in numerical approximations used in this method with discontinuous coefficients, often the largest numerical errors are

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Nomenclature

а	thermal diffusivity (m ² /s)		
C_p	specific heat capacity (J/kg K)		
Ĺ	latent heat (J/kg)		
Т	temperature ($^{\circ}C$)		
t	time (s)		
<i>x</i> , <i>z</i>	Cartesian coordinates		
Greek symbols			
λ	effective thermal conductivity (W/m K)		
ϕ	porosity		
Subscripts			
f	fusion		
i	initial		
j	layer number		
1	unfrozen		
mov	moving boundary		
S	frozen		

introduced in a neighborhood of the discontinuities particularly for phase change in geometry complexity as well as boundary condition.

The troublesome numerical errors in conventional method are effectively reduced if the grid generation and solution procedure are separated with the discontinuities and special formulas are used to incorporate the jump conditions directly into the numerical model. This is the main idea behind this work considering moving boundary as a parameter.

To create a computational grid in body-fitted coordinates, two basic steps required: (1) define an origin point and (2) specify the distribution (number and spacing) of grid nodes along the edges of the geometric regions. The automatic grid generator then takes over, and using an algebraic technique known as transfinite interpolation, creates a grid that simultaneously matches the edge node prescription and conforms to the irregular edges of the body-fitted geometry. Grid generation by algebraic methods produces high-quality numerical grids and allow for the very efficient integration of the thermal-flow field physics. Considering grid optimization, the designed grid optimization-algorithm improves upon the transfinite interpolation method by carrying the grid generation process one step further. It uses automatically generated grid as an initial approximation to a higher quality grid system derived utilizing the technique of PDE grid generation. This technique offers advantages over purely algebraic methods:

- Good control over the skewness and spacing of the derived grid on surface interiors, while simultaneously allowing complete control over the grid spacing (node distribution) on surface edges as well as moving boundary.
- An ability to produce unique, stable, and smooth grid distributions free of interior maxima or minima (inflection points) in body-fitted coordinates.

Parabolic grid generation works well with irregularly shaped geometries and can produce grids that are highly conformal with the edges of individual computational surfaces. The means for grid generation should not be dictated by the limitations of a given specific field solution procedure and conversely the method that determines the field should accept as input an arbitrary set of coordinate points which constitutes the grid. In general, of course, these two operations can never be totally independent because the logistic structure of the information, the location of outer boundaries, the nature of coordinate and the types of grid singularities are items that have to be coordinated closely between the field solver and the grid generator (Eriksson, 1982).

Grid generation for multi-dimensional geometries using transfinite interpolation functions was studied by Coons (1967), Cook (1974), Gordon and Hall (1973), and Ettouney and Brown (1983) successfully modeled slightly non-planar interfaces by using an algebraic grid generation system where the interface was described in terms of univariate function.

Although grid generation is the core of most numerical algorithms for phase change problems or non-phase change problem, little effort has been reported on phase change problems, particularly the problem which couples the grid generation algorithm with the heat transport equations.

The present paper introduces the novel numerical approaches for freezing problems, which extend the range of initial condition and boundary conditions in case of multiple heat sources with different temperature that can be covered. They will also permit a continuous determination of the multi-dimensional freezing front and indicate the internal temperature distribution with a greater degree of boundary complexity and offers the highest overall accuracies and smooth grid point distribution. Numerically, for generating a boundary/interface fitted coordinate system, structured grids are initialized using transfinite interpolation algebraic techniques and the quality of structured grids can be significantly improved by applying parabolic-PDE methods. These methods iteratively solve unsteady conduction's equation together with moving boundary condition during the freezing process considering conduction as the only mode of heat transfer in both the unfrozen layer and the frozen layer.

2. Modeling formulation

The two-dimensional system illustrated schematically in Fig. 1 is considered. Initially, the walls are all insulated and the rectangular cavity is filled with a porous medium (PM) consisting of the glass beads and phase change material (PCM) in the liquid state (water), both at the fusion temperature T_f (0 °C). Multiple heat sources with specified temperature (T_L) are located at the top wall. At time t=0, the freezing process downwardly begins. The applicable differential equations for two-dimensional heat flow with constant thermal properties for the unfrozen and frozen layers are, respectively:

$$\frac{\partial T_1}{\partial t} = a_1 \left(\frac{\partial^2 T_1}{\partial x^2} + \frac{\partial^2 T_1}{\partial z^2} \right) + \left(\frac{\partial T_1}{\partial z} \right) \frac{\mathrm{d}z}{\mathrm{d}t} \tag{1}$$

$$\frac{\partial T_{\rm s}}{\partial t} = a_{\rm s} \left(\frac{\partial^2 T_{\rm s}}{\partial x^2} + \frac{\partial^2 T_{\rm s}}{\partial z^2} \right) + \left(\frac{\partial T_{\rm s}}{\partial z} \right) \frac{\mathrm{d}z}{\mathrm{d}t} \tag{2}$$

where the last terms of Eqs. (1) and (2) result from a coordinate transformation attached to the moving boundary. In the unfrozen layer, if internal natural convection can be neglected because the presence of glass beads minimizes the effect of natural convection current.

Eqs. (1) and (2) are based on the following assumptions.

- (1) the temperature field can be assumed to be two-dimensional,
- (2) the thermal equilibrium exists between PCM and PM; this is possible when the porous matrix has a little larger thermal conductivity than the PCM, and the interphase heat transfer can be properly neglected,
- (3) properties of PM are isotropic.

The boundary conditions of Eqs. (1) and (2) are:

(a) the localized freezing condition at the top horizontal wall, the multiple heat sources with specified temperature (T_L) are applied:

$$x_{ll} \le x_l \le x_{lr}: \quad T = T_L, \qquad x_{cl} \le x_c \le x_{cr}: \quad T = T_L, \qquad x_{rl} \le x_r \le x_{rr}: \quad T = T_L$$
(3)

where various subscripts of x denote the regions for applying heat sources with strip length of 10 mm in each region. In addition, for the case of the freezing of water-saturated porous media in a rectangular cavity subjected to multiple heat sources with different temperature, the temperatures (T_L) in each region are different.

(b) adiabatic condition: the walls are all insulated

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial z} = 0$$

(c) moving boundary condition:



Fig. 1. Physical model.

(4)

The moving boundary condition (Stefan condition), which is obtained from a consideration of the energy balance at the interface between the frozen layer and unfrozen layer provides following equation:

$$\left(\lambda_{s}\frac{\partial T_{s}}{\partial z} - \lambda_{1}\frac{\partial T_{1}}{\partial z}\right)\left[1 + \left(\frac{\partial z_{mov}}{\partial x}\right)^{2}\right] = \rho_{s}L_{s}\frac{\partial z_{mov}}{\partial t}$$
(5)

where $\partial z_{mov}/\partial_t$ is the velocity of fusion front or freezing front, and L_s is the latent heat of fusion. To avoid changes in the physical dimensions as the freezing front progresses, $\rho_s = \rho_1$ will be specified. In this study, the thermal conductivity, λ_1 is bulk-average value for the glass beads and water and λ_s is bulk-average value for the glass beads and ice.

3. Grid generation technique

Generally, two types of structured grid generation are currently in use. They are algebraic method, i.e., transfinite interpolation method and PDE methods. Transfinite interpolation provides a relatively easy way of obtaining an initial grid that can be refined and smoothed by other techniques, whether algebraic, PDE method.

The main idea behind this work, prior to generation of grids by PDE methods, it is preferable to obtain first preliminary grids using the algebraic method, i.e., transfinite interpolation technique. The combined transfinite interpolation and PDE method is used to achieve a smoother grids point distribution and boundary point discontinuities are smoothed out in the interior domain.

3.1. Transfinite interpolation (TFI)

The method of constructing a two-dimensional boundary-conforming grid for a system is a direct algebraic approach based on the concept of TFI. In this method, no partial differential equations are solved to obtain the curvilinear coordinates, and the same system is used for the entire domain. The algebraic technique can be easier to construct than PDE methods, and gives also easier control over grid characteristics such as orthogonality and grid point spacing. However, this method is sometime criticized for allowing discontinuities on the boundary to propagate into the interior and for not generating grids as smooth as those generated by PDE method.

The technique used for transfinite interpolation here is a significant extension of the original formulation by Gordon and Hall (1973). It possible to initially the generate global grid system with geometry specifications only on the outer boundaries of the computation domain and yet to obtain a high degree of local control.

Fig. 2 illustrates the present method of constructing a two-dimensional boundary-conforming grid for a system, which is a direct algebraic approach based on the concept of transfinite or multivariate interpolation. It is possible to initially generate global single plane transformations with geometry specifications only on outer boundaries of the computational domain.

Let f(u, w) = (x(u, w), z(u, w)) denote a vector-valued function of two parameters u, w defined on the region $u_1 \le u \le u_{\text{max}}$, $w_1 \le w \le u_{\text{max}}$. This function is not known throughout the region, only on certain planes (Fig. 2). The transfinite interpolation procedure then gives the interpolation function $f_{(u,w)}$ by the recursive algorithm:

$$f_{(u,w)}^{(1)} = A_{1(u)}f_{(1,w)} + A_{2(u)}f_{(u_{\max},w)}, \qquad f_{(u,w)} = f_{(u,w)}^{(1)} + B_{1(w)}[f_{(u,1)} - f_{(u,1)}^{(1)}] + B_{2(w)}[f_{(u,w_{\max})} - f_{(u,w_{\max})}^{(1)}]$$
(6)



Fig. 2. The parametric domain with $f_{(u,w)}$ specified on planes of constant u, w.

where $A_{1(u)}, A_{2(u)}, B_{1(w)}$ and $B_{2(w)}$ are defined by the set of univariate blending functions, which only have to satisfy the conditions:

$$A_{1(1)} = 1, \quad A_{1(u_{\max})} = 0; \qquad A_{2(1)} = 0, \quad A_{2(u_{\max})} = 1; \qquad B_{1(1)} = 1, \quad B_{1(w_{\max})} = 0; \qquad B_{2(1)} = 0, \quad B_{2(w_{\max})} = 1$$
(7)

Further, the general form in algebraic equations can be defined as:

$$A_{1(u)} = \frac{u_{\max} - u}{u_{\max} - 1}, \quad A_{2(u)} = 1 - A_{1(u)}; \qquad B_{1(w)} = \frac{w_{\max} - w}{w_{\max} - 1}, \quad B_{2(w)} = 1 - B_{1(w)}$$
(8)

The grid motion defined from a moving boundary motion is modeled using a Stefan condition (Eq. (5)) with a transfinite mapping technique.

The boundary fitted grid generation mapping discussed in this section forms the basis for the interface-tracking mapping. However, the mapping now must match the interface curve on the interior of physical domain in addition to fitting the outer physical boundary. In addition, the system must be adaptive since the grid lines must change to follow the deforming interface while maintaining as much smoothness and orthogonality as possible.

3.2. PDE method

In the proposed grid generation mapping, all grids discussed and displayed have been couched in terms of finite difference formulation, with the understanding that whatever non-uniform grid exists in the physical space, there exists a transformation, which will recast it as a uniform rectangular grid in the computational space. The finite difference calculations are then made over this uniform grid in the computational space, after which the field results are transferred directly back to the corresponding points in the physical space. The purpose of generating a smooth grid that conforms to physical boundaries of problem is, of course, to solve the partial differential equations specified in the problem by finite difference scheme, capable of handling general non-orthogonal curvilinear coordinates.

Fig. 1 show that, as freezing proceeds, the freezing front denoted by z_{mov} is formed. Due to the existence of this freezing front, the frozen and unfrozen domains are irregular and time dependent. To avoid this difficulty, a curvilinear system of coordinates is used to transform the physical domain into rectangular region for the computational domain.

It is convenient to introduce a general curvilinear coordinate system as follows (John & Anderson, 1995):

$$x = x(\xi, \eta), \ z = z(\xi, \eta) \text{ or } \xi = \xi(x, z), \ \eta = \eta(x, z)$$
(9)

The moving boundaries are immobilized in the dimensionless (ξ , η) coordinate for all times. With the details omitted, the transformation of Eqs. (1), (2) and (5) can be written respectively as:

$$\frac{\partial T_{l}}{\partial t} = \frac{a_{l}}{J^{2}} \left(\alpha \frac{\partial^{2} T_{l}}{\partial \xi^{2}} - 2\beta \frac{\partial^{2} T_{j}}{\partial \xi \partial \eta} + \gamma \frac{\partial^{2} T_{l}}{\partial \eta^{2}} \right) + \frac{a_{l}}{J^{3}} \left[\left(\alpha \frac{\partial^{2} x}{\partial \xi^{2}} \right) \left(z_{\xi} \frac{\partial T_{l}}{\partial \eta} - z_{\eta} \frac{\partial T_{l}}{\partial \xi} \right) + \alpha \frac{\partial^{2} z}{\partial \xi^{2}} - 2\beta \frac{\partial^{2} z}{\partial \xi \partial \eta} + \gamma \frac{\partial^{2} z}{\partial \eta^{2}} \left(-x_{\xi} \frac{\partial T_{l}}{\partial \eta} \right) \right] + \frac{1}{J} \left(x_{\xi} \frac{\partial T_{l}}{\partial \eta} \right) \frac{dz}{dt} \tag{10}$$

$$\frac{\partial T_{s}}{\partial t} = \frac{a_{s}}{J^{2}} \left(\alpha \frac{\partial^{2} T_{s}}{\partial \xi^{2}} - 2\beta \frac{\partial^{2} T_{s}}{\partial \xi \partial \eta} + \gamma \frac{\partial^{2} T_{s}}{\partial \eta^{2}} \right) + \frac{a_{s}}{J^{3}} \left[\left(\alpha \frac{\partial^{2} x}{\partial \xi^{2}} \right) \left(z_{\xi} \frac{\partial T_{s}}{\partial \eta} - z_{\eta} \frac{\partial T_{s}}{\partial \xi} \right) + \alpha \frac{\partial^{2} z}{\partial \xi^{2}} - 2\beta \frac{\partial^{2} z}{\partial \xi \partial \eta} + \gamma \frac{\partial^{2} z}{\partial \eta^{2}} \left(-x_{\xi} \frac{\partial T_{s}}{\partial \eta} \right) \right] + \frac{1}{J} \left(x_{\xi} \frac{\partial T_{s}}{\partial \eta} \right) \frac{dz}{dt}$$
(11)

$$\left\{\lambda_{s}\frac{1}{J}\left(x_{\xi}\frac{\partial T_{s}}{\partial\eta}\right) - \lambda_{l}\frac{1}{J}\left(x_{\xi}\frac{\partial T_{l}}{\partial\eta}\right)\right\}\left\{1 + \left(\frac{1}{J}\left[z_{\eta}\frac{\partial z_{mov}}{\partial\xi} - z_{\xi}\frac{\partial z_{mov}}{\partial\eta}\right]\right)^{2}\right\} = \rho_{s}L_{s}\frac{\partial z_{mov}}{\partial t}$$
(12)

where $J = x_{\xi}z_{\eta} - x_{\eta}z_{\xi}$, $\alpha = x_{\eta}^2 + z_{\eta}^2$, $\beta = x_{\xi}x_{\eta} + z_{\xi}z_{\eta}$, $\gamma = x_{\xi}^2 + z_{\xi}^2$, and x_{ξ} , x_{η} , z_{ξ} and z_{η} denote partial derivatives, J the Jacobian, β , α , γ the geometric factors and η , ξ are the transformed coordinates.

 Table 1

 Thermal property of the unfrozen layer and frozen layer

Properties	Unfrozen layer	Frozen layer
$\frac{1}{\rho (\text{kg/m}^3)}$	1942.0	1910.0
$\lambda (W/m K)$	0.210 × 10 °	0.605 × 10 ° 1.480
$C_p (J/\text{kg K})$	2.099×10^{3}	1.281×10^{3}

4. Solution method

It is known that the inherent difficulties in the conventional numerical methods (pure parabolic grid generators) for freezing problems suggest the use of combined transfinite interpolation and PDE methods is most instances. Although conventional numerical methods can be used to obtain satisfactory results, problems of large time consumption and control functions that are often difficult to determine, are involved. Therefore, the new method presented in this paper is generally applicable, and offers the highest overall accuracies and smooth grid point distribution. In addition, the boundary point discontinuities are smoothed out in the interior domain and orthogonality at boundaries can be maintained.

In this study, in order to initiate numerical simulation, a very thin layer of freeze with a constant thickness $z_{mov(0)}$ was assumed to be present. This initial condition is obtained from the Stefan solution in the frozen layer and a linear temperature distribution in the unfrozen layer. Tests revealed that the influence of $z_{mov(0)}$ could be neglected as $z_{mov(0)}$ was sufficiently small. The transient heat equations (Eqs. (10) and (11)) and the Stefan condition (Eq. (12)) are solved by using finite difference method using parameter values obtained from Table 1. A system of non-linear equations results whereby each equation for the internal nodes can be cast into a numerical discretization:

Transient heat equation for unfrozen layer:

$$\begin{split} T_{l}^{n+1}(k,i) &= \left(\frac{1}{1+(2a_{l}\Delta t/(J^{2}(k,i))((\alpha(k,i)/\Delta\zeta\Delta\zeta) + (\gamma(k,i)/\Delta\eta\Delta\eta))))}\right) \\ &\times \left(T_{l}^{n}(k,i) + \frac{a_{l}\Delta t}{J^{2}(k,i)} \left(\alpha(k,i) \frac{T_{l}^{n+1}(k,i+1) + T_{l}^{n+1}(k,i-1)}{\Delta\zeta\Delta\zeta}\right) \\ &- 2\beta(k,i) \left(\left(\frac{T_{l}^{n+1}(k+1,i+1) - T_{l}^{n+1}(k-1,i+1)}{2\Delta\eta}\right)\right) \\ &- \left(\frac{T_{l}^{n+1}(k+1,i-1) - T_{l}^{n+1}(k-1,i-1)}{2\Delta\zeta}\right)\right) / 2\Delta\zeta + \gamma(k,i) \left(\frac{T_{l}^{n+1}(k+1,i) + T_{l}^{n+1}(k-1,i)}{\Delta\eta\Delta\eta}\right)\right) \\ &+ \frac{a_{l}\Delta t}{J^{3}(k,i)} \left(\left(\alpha(k,i) \frac{X(k,i+1) - 2X(k,i) + X(k,i-1)}{\Delta\zeta\Delta\zeta}\right) \times \left(\left(\frac{Z(k,i+1) - Z(k,i-1)}{2\Delta\zeta}\right)\right) \\ &\times \left(\frac{T_{l}^{n+1}(k+1,i) - T_{l}^{n+1}(k-1,i)}{2\Delta\eta}\right) - \left(\frac{Z(k+1,i) - Z(k-1,i)}{2\Delta\eta}\right) \\ &\times \left(\frac{T_{l}^{n+1}(k,i+1) - T_{l}^{n+1}(k,i-1)}{2\Delta\zeta}\right) + \alpha(k,i) \left(\frac{Z(k,i+1) - 2Z(k,i) + Z(k,i-1)}{\Delta\zeta\Delta\zeta}\right) \\ &- 2\beta(k,i) \left(\left(\frac{Z(k+1,i+1) - Z(k-1,i+1)}{2\Delta\eta}\right) - \left(\frac{Z(k+1,i-1) - Z(k-1,i-1)}{2\Delta\eta}\right)\right) / 2\Delta\zeta \\ &+ \gamma(k,i) \left(\frac{Z(k+1,i) - 2Z(k,i) + Z(k-1,i)}{\Delta\eta\Delta\eta}\right) \times \left(-\frac{(X(k,i+1) - X(k,i-1))}{2\Delta\zeta}\right) \\ &\times \left(\frac{T_{l}^{n+1}(k+1,i) - T_{l}^{n+1}(k-1,i)}{2\Delta\eta}\right) + \frac{1}{J(k,i)} \left(\frac{X(k,i+1) - X(k,i-1)}{2\Delta\zeta}\right) \end{split}$$
(13)



Fig. 3. Strategy for calculation.



Fig. 4. Validation testes for a planar freezing front in a rectangular phase-change Slab.



Fig. 5. Comparison of experimental data and simulated freezing front from present numerical study.



Fig. 6. Grid simulating the deformation of an interface (subjected to multiple constant temperature heat sources): freezing time of (a) 60 s, (b) 120 s, and (c) 180 s.

Transient heat equation for frozen layer:

$$T_{s}^{n+1}(k,i) = \left(\frac{1}{1 + (2a_{s}\Delta t/(J^{2}(k,i))((\alpha(k,i)/\Delta\zeta\Delta\zeta) + (\gamma(k,i)/\Delta\eta\Delta\eta))))}\right)$$
$$\times \left(T_{s}^{n}(k,i) + \frac{a_{s}\Delta t}{J^{2}(k,i)} \times \left(\alpha(k,i)\frac{T_{s}^{n+1}(k,i+1) + T_{s}^{n+1}(k,i-1)}{\Delta\zeta\Delta\zeta}\right)\right)$$

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$$\begin{split} &-2\beta(k,i)\left(\left(\frac{T_s^{n+1}(k+1,i+1)-T_s^{n+1}(k-1,i+1)}{2\Delta\eta}\right)\right.\\ &-\left(\frac{T_s^{n+1}(k+1,i-1)-T_s^{n+1}(k-1,i-1)}{2\Delta\zeta}\right)\right)/2\Delta\zeta+\gamma(k,i)\left(\frac{T_s^{n+1}(k+1,i)+T_s^{n+1}(k-1,i)}{\Delta\eta\Delta\eta}\right)\right)\\ &+\frac{a_s\Delta t}{J^3(k,i)}\left(\left(\alpha(k,i)\frac{X(k,i+1)-2X(k,i)+X(k,i-1)}{\Delta\zeta\Delta\zeta}\right)\times\left(\left(\frac{Z(k,i+1)-Z(k,i-1)}{2\Delta\zeta}\right)\right)\right.\\ &\times\left(\frac{T_s^{n+1}(k+1,i)-T_s^{n+1}(k-1,i)}{2\Delta\eta}\right)-\left(\frac{Z(k+1,i)-Z(k-1,i)}{2\Delta\eta}\right)\\ &\times\left(\frac{T_s^{n+1}(k,i+1)-T_s^{n+1}(k,i-1)}{2\Delta\zeta}\right)\right)+\alpha(k,i)\left(\frac{Z(k,i+1)-2Z(k,i)+Z(k,i-1)}{2\Delta\eta}\right)\\ &-2\beta(k,i)\left(\left(\frac{Z(k+1,i+1)-Z(k-1,i+1)}{2\Delta\eta}\right)-\left(\frac{Z(k+1,i-1)-Z(k-1,i-1)}{2\Delta\zeta}\right)\right)/2\Delta\zeta\\ &+\gamma(k,i)\left(\frac{Z(k+1,i)-2Z(k,i)+Z(k-1,i)}{2\Delta\eta}\right)\times\left(-\frac{(X(k,i+1)-X(k,i-1))}{2\Delta\zeta}\right)\\ &\times\left(\frac{T_s^{n+1}(k+1,i)-T_s^{n+1}(k-1,i)}{2\Delta\eta}\right)+\frac{1}{J(k,i)}\left(\frac{X(k,i+1)-X(k,i-1)}{2\Delta\zeta}\right)\\ &\times\left(\frac{T_s^{n+1}(k+1,i)-T_s^{n+1}(k-1,i)}{2\Delta\eta}\right)\timesdz(k,i)\right) \end{split}$$

(14)



Fig. 7. The simulations of temperature distribution ($^{\circ}$ C) within rectangular phase change slab (subjected to multiple constant temperature heat sources): freezing time of (a) 60 s, (b) 120 s, and (c) 180 s.

Stefan condition:

$$Z^{n+1}(k,i) = Z^{n}(k,i) + \frac{\Delta T}{\rho_{s}L_{s}} \times \left[\left(\frac{\lambda_{s}}{J(k-1,i)} \times \left(\frac{X(k-1,i+1) - X(k-1,i-1)}{2\Delta\zeta} \right) \right) \\ \times \left(\frac{3T(k,i) - 4T(k-1,i) + T(k-2,i)}{2\Delta\eta} \right) - \frac{\lambda_{1}}{J(k+1,i)} \times \left(\frac{X(k+1,i+1) - X(k+1,i-1)}{2\Delta\zeta} \right) \right) \\ \times \left(\frac{-3T(k,i) + 4T(k+1,i) - T(k+2,i)}{2\Delta\eta} \right) \right) \\ \times \left(1 + \left(\left(\frac{Z(k+1,i) - Z(k-1,i)}{2\Delta\eta} \right) \times \left(\frac{Z^{n}(k+1,i) - Z^{n}(k-1,i)}{2\Delta\zeta} \right) \right) \\ - \left(\frac{Z(k,i+1) - Z(k,i-1)}{2\Delta\zeta} \right) \times \left(\frac{Z^{n}(k,i+1) - Z^{n}(k,i-1)}{2\Delta\eta} \right) \right)^{2} \right) \right]$$
(15)

The details of computational schemes and strategy for solving the combined transfinite interpolation functions (Eqs. (6)–(9)) and PDE (Eqs. (13)–(15)) are illustrated in Fig. 3.

5. Experiment

The freezing experiments are performed in a rectangular test cell filled with a porous medium (porosity, $\phi = 0.38$) with inside dimensions of 10 cm in length (*x*), 5 cm in height (*z*) and 2.5 cm in depth (*y*). The partial horizontal top wall and bottom wall and



Fig. 8. Grid simulating the deformation of an interface (subjected to multiple heat sources with different temperature): freezing time of (a) 60 s, (b) 120 s, and (c) 180 s.

the vertical front and back walls are made of acrylic resin. The entire test cell is covered with 8 cm thick Styrofoam on all sides to minimize the effect of heat losses and condensation of moisture at the walls. The partial top wall, which serves as a constant temperature heat source, is multi-pass heat exchanger. Heat exchanger is connected through a valve system to constant temperature bath where the liquid nitrogen is used as the cooling medium. Thirty thermocouples with diameter of 0.15 mm are placed at interval of 10 mm throughout the axis of a sample (x and z planes). These thermocouples are connected to data-logger and computer through which the temperatures could be measured and stored at preselected time intervals. The positions of freezing front in the sample are determined by interpolating the fusion temperature from the thermocouples reading.

The uncertainty in the results might come from the variations in humidity, room temperature and human error. The calculated uncertainty associated with temperature is less than 2.75%. The calculated uncertainties in all tests are less than 2.85%.

6. Results and discussion

Numerical results are obtained for phase change in a rectangular cavity filled with a porous medium. The calculations are performed under the following conditions:

- (1) The time step of dt = 0.1 [s] is used for the computation of temperature field and location of freezing front.
- (2) The number of nodes is N = 120 (width) $\times 100$ (depth).
- (3) Iterations are carried out until relative errors of 10^{-8} are reached.

In order to verify the accuracy of the present numerical algorithm, it is validated by performing simulations for a planar freezing front in a water layer with a dimension of $10 \text{ cm}(x) \times 5 \text{ cm}(z)$. Initially, the temperature of 0°C is assigned throughout the layer.



Fig. 9. Grid simulating the deformation of an interface (subjected to multiple heat sources with different temperature): freezing time of (a) 60 s, (b) 120 s, and (c) 180 s.

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Thereafter, one constant temperature heat source ($T_L = -40 \,^{\circ}$ C) with strip length of 10 mm is imposed on the top wall. The calculated front location is based on the thermal properties of ice and water. The results are then compared with analytical solution for the freezing of water layer at the same condition. Fig. 4 clearly shows a good agreement between simulated and analytical solutions. Therefore, the present method can yield accurate solutions.

Fig. 5 shows the measured and simulated results of the freezing front during freezing of water-saturated porous media in a rectangular cavity with a dimension of 10 cm (x) × 5 cm (z) and initial temperature of 0 °C. In this comparison, one constant temperature heat source, $T_L = -40$ °C, is applied. The observation of the freezing front depicted from the figure reveals that the simulated results and experimental results are qualitatively consistent. However, the experimental data is significantly lower than that simulated results. Discrepancy may be attributed to heat loss and non-uniform heating effect along the surface of supplied load. Numerically, the discrepancy may be attributed to uncertainties in the thermal and physical properties data. In addition, the source of the discrepancy may be attributed to natural convection effect in liquid.

6.1. Freezing process with multiple constant temperature heat sources

The purpose of this subsection is to illustrate the efficiency of the grid generation system during the freezing of water-saturated porous media in a rectangular cavity with a dimension of $12 \text{ cm} (x) \times 5 \text{ cm} (z)$ (porosity, $\phi = 0.38$) subjected to multiple constant temperature heat sources (three heat sources with strip length of 10 mm in each region). Fig. 6(a) through (c) show grids that fit curves that are typical of shapes seen during deformation of an interface with respect to elapsed times. The grid generation corresponds to the initial temperature of 0 °C and applied boundary condition ($T_L = -40$ °C) given by Eq. (3). It can be seen how freezing fronts progress with respect to elapsed times. During the initial stages of freezing the shape of the interface in each region becomes a small semi-circular shape as the freezing front moves further away from the fixed boundaries indicating principally two-dimensional heat flow. As later times, the curve on the interface gradually flattens indicating the two-dimensional effect. In all figures, it is found that the grid is able to maintain a significant amount of orthogonality and smoothness both within the interior and along the boundary



Fig. 10. The simulations of temperature distribution ($^{\circ}$ C) (subjected to multiple heat sources with different temperature): freezing time of (a) 60 s, (b) 120 s, and (c) 180 s.

as the grid points redistribute themselves to follow the interface. These results show the efficiency of the present method for the moving boundary problem.

The simulations of temperature distribution within rectangular cavity filled with porous media in the vertical plane (x-z) corresponding to grid simulating the deformation of an interface (Fig. 6(a)–(c)), are shown in Fig. 7(a)–(c). Since the present work is to couple the grid generation algorithm with the transport equations, the thermal analysis during freezing process will be discussed as follows. When multiple constant temperature heat sources are applied during localized freezing process, heat is conducted from the hotter region in unfrozen layer to the cooler region in frozen layer. At the initial stages of freezing, the freezing fronts exhibit to be a small semi-circular shape indicating principally two-dimensional heat conduction as shown in Fig. 7(a). Later, the shape of interface becomes larger semi-circular shape as the freezing front moves further away from the fixed boundary as shown in Fig. 7(b) and (c). However, as the freezing process persists, the freezing rate progresses slowly. This is because most of heat conduction takes place the leading edge of frozen layer (freeze layer), which is located further from unfrozen layer. Consequently, small amount of heat can conduct to the frozen layer due to the freeze region acting as an insulator and causing a freezing front to slowly move with respect to elapsed times. Considering the shapes of the freezing front with respect to elapsed times, each freezing region of the rectangular cavity shows signs of freezing, while the outer edge displays no obvious sign of freezing indicating that the temperature does not fall below 0 °C. Nevertheless, at the long stages of freezing, the leading edge of applied boundary condition displays sign of freezing continuously and the spreading of the freeze in the both x-z directions (semi-circular shape) is clearly shown.

6.2. Freezing process with multiple heat sources with different temperature

The following discussion refers to case that the freezing of water-saturated porous media in a rectangular cavity with a dimension of 12 cm (x) × 5 cm (z) (porosity, $\phi = 0.38$) subjected to multiple heat sources with different temperature. The grid generation for the applied boundary condition with different temperature in two cases, namely, $T_{L,l} = -80$ °C, $T_{L,c} = -40$ °C, $T_{L,r} = -80$ °C and



Fig. 11. The simulations of temperature distribution ($^{\circ}$ C) (subjected to multiple heat sources with different temperature): freezing time of (a) 60 s, (b) 120 s, and (c) 180 s.

 $T_{L,l} = -40 \text{ °C}$, $T_{L,c} = -60 \text{ °C}$, $T_{L,r} = -80 \text{ °C}$, are shown in Figs. 8(a)–(c) and 9(a)–(c), respectively. The simulations of temperature distribution corresponding these grid generations (Figs. 8 and 9), are also shown in Figs. 10(a)–(c) and 11(a)–(c), respectively. As similarly mentioned in previous subsection, the simulated results show the reasonable trends of freezing phenomena at specified freezing conditions.

This study shows the capability of the present method to correctly handle the phase change problem with highly complex moving boundaries condition. With further quantitative validation of the present method, this method can be used as a tool for investigating in detail this particular freezing of phase change slab at a fundamental level.

7. Conclusions

Mesh quality has the largest impact on solution quality. A high quality mesh increases the accuracy of the computational flow solution and improves convergence. Therefore, it is important to provide tools for obtaining and improving a mesh.

In this study, the freezing of water-saturated porous media in a rectangular cavity subjected to multiple heat sources with different temperature has been investigated numerically. A generalized mathematical model and an effective calculation procedure is proposed. A preliminary case study indicates the successful implementation of the numerical procedure. A two-dimensional freezing model is then validated against available analytical solutions and experimental results and subsequently used as a tool for efficient computational prototyping. Simulated results are in good agreement with available analytical solution and experimental results. The successful comparison with analytical solution and experiments should give confidence in the proposed mathematical treatment, and encourage the acceptance of this method as useful tool for exploring practical problems.

The next phase, which has already begun, is to couple the grid generation algorithm with the completing transport equations that determine the moving boundary front and buoyancy-driven convection in the unfrozen layer (liquid). Moreover, some experimental studies will be performed to validate numerical results.

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Appendix A

In this section, we will derive a transformation model of the governing differential equations for using in the numerical calculation. The details are shown as below:

A.1. General transformation of the first and second derivatives

Considering the first derivative of any parameters can be written as:

$$\frac{\partial}{\partial x} = \frac{1}{J} \left(z_{\eta} \frac{\partial}{\partial \xi} - z_{\xi} \frac{\partial}{\partial \eta} \right), \qquad \frac{\partial}{\partial z} = \frac{1}{J} \left(x_{\xi} \frac{\partial}{\partial \eta} - x_{\eta} \frac{\partial}{\partial \xi} \right)$$
(A.1)

where J is Jacobian, it can be written as:

$$J = x_{\xi} z_{\eta} - x_{\eta} z_{\xi} \tag{A.2}$$

$$x_{\xi} = \frac{\partial x}{\partial \xi} \tag{A.3}$$

Considering the second derivative of any parameters, we will establish the second derivative of Laplace equation of parameter A where Eqs. (A.1)–(A.3) are related:

$$\nabla^{2}A = \left(\frac{\partial^{2}}{\partial x^{2}} + \frac{\partial^{2}}{\partial z^{2}}\right)A = \frac{1}{J^{2}}\left(\alpha\frac{\partial^{2}A}{\partial\xi^{2}} - 2\beta\frac{\partial^{2}A}{\partial\xi\partial\eta} + \gamma\frac{\partial^{2}A}{\partial\eta^{2}}\right) \\ + \frac{1}{J^{3}}\left[\left(\alpha x_{\xi\xi} - 2\beta x_{\xi\eta} + \gamma x_{\eta\eta}\right)\left(z_{\xi}\frac{\partial A}{\partial\eta} - z_{\eta}\frac{\partial A}{\partial\xi}\right) + \left(\alpha z_{\xi\xi} - 2\beta z_{\xi\eta} + \gamma z_{\eta\eta}\right)\left(x_{\eta}\frac{\partial A}{\partial\xi} - x_{\xi}\frac{\partial A}{\partial\eta}\right)\right]$$
(A.4)

where

$$\alpha = x_{\eta}^{2} + z_{\eta}^{2}, \qquad \beta = x_{\xi}x_{\eta} + z_{\xi}z_{\eta}, \qquad \gamma = x_{\xi}^{2} + z_{\xi}^{2}$$
(A.5)

 x_{ξ} , x_{ξ} , z_{ξ} and z_{η} denote partial derivatives, β , α , γ are the geometric factors and η , ξ are the transformed coordinates. The related parameter can be defined as:

$$x = x(\xi, \eta), \ z = z(\xi, \eta) \quad \text{or} \quad \xi = \xi(x, z), \ \eta = \eta(x, z)$$

arrow
$$x = x(\xi), \ z = z(\xi, \eta) \quad \text{or} \quad \xi = \xi(x), \ \eta = \eta(x, z)$$

$$\therefore \ x_{\eta} = \frac{\partial x}{\partial \eta} = 0 \quad \text{or} \quad \xi_{x} = \frac{\partial \xi}{\partial x} = 0$$

(A.6)

Corresponding to the Eq. (A.6), the first derivative of any parameters (Eq. (A.1)) can be rewritten as:

$$\frac{\partial}{\partial x} = \frac{1}{J} \left(z_{\eta} \frac{\partial}{\partial \xi} - z_{\xi} \frac{\partial}{\partial \eta} \right), \qquad \frac{\partial}{\partial z} = \frac{1}{J} \left(x_{\xi} \frac{\partial}{\partial \eta} - x_{\eta} \frac{\partial}{\partial \xi} \right)$$
arrow

$$\frac{\partial}{\partial x} = \frac{1}{J} \left(z_{\eta} \frac{\partial}{\partial \xi} - z_{\xi} \frac{\partial}{\partial \eta} \right), \qquad \frac{\partial}{\partial z} = \frac{1}{J} \left(x_{\xi} \frac{\partial}{\partial \eta} \right)$$
(A.7)

where

$$J = x_{\xi} z_{\eta} - x_{\eta} z_{\xi}$$

arrow
$$J = x_{\xi} z_{\eta}$$
 (A.8)

The second derivative of any parameters (Eqs. (A.4)–(A.6)) can be also rewritten as:

$$\nabla^{2}A = \frac{1}{J^{2}} \left(\alpha \frac{\partial^{2}A}{\partial\xi^{2}} - 2\beta \frac{\partial^{2}A}{\partial\xi\partial\eta} + \gamma \frac{\partial^{2}A}{\partial\eta^{2}} \right) + \frac{1}{J^{3}} \left[(\alpha x_{\xi\xi} - 2\beta x_{\xi\eta} + \gamma x_{\eta\eta}) \left(z_{\xi} \frac{\partial A}{\partial\eta} - z_{\eta} \frac{\partial A}{\partial\xi} \right) + (\alpha z_{\xi\xi} - 2\beta z_{\xi\eta} + \gamma z_{\eta\eta}) \left(x_{\eta} \frac{\partial A}{\partial\xi} - x_{\xi} \frac{\partial A}{\partial\eta} \right) \right]$$
(A.9)

arrow

$$\nabla^2 A = \frac{1}{J^2} \left(\alpha \frac{\partial^2 A}{\partial \xi^2} - 2\beta \frac{\partial^2 A}{\partial \xi \partial \eta} + \gamma \frac{\partial^2 A}{\partial \eta^2} \right) + \frac{1}{J^3} \left[(\alpha x_{\xi\xi}) \left(z_{\xi} \frac{\partial A}{\partial \eta} - z_{\eta} \frac{\partial A}{\partial \xi} \right) + (\alpha z_{\xi\xi} - 2\beta z_{\xi\eta} + \gamma z_{\eta\eta}) \left(-x_{\xi} \frac{\partial A}{\partial \eta} \right) \right]$$

where

$$\alpha = x_{\eta}^{2} + z_{\eta}^{2}, \quad \beta = x_{\xi}x_{\eta} + z_{\xi}z_{\eta}, \quad \gamma = x_{\xi}^{2} + z_{\xi}^{2}$$

arrow
$$\alpha = z_{\eta}^{2}, \quad \beta = z_{\xi}z_{\eta}, \quad \gamma = x_{\xi}^{2} + z_{\xi}^{2}$$
(A.10)

A.2. The transformation of thermal model

After some mathematical manipulations (Eq. (A.7), (A.9), (1), (2) and (5)), a transformation model of the governing differential equations become:

$$\frac{\partial T_{1}}{\partial t} = \frac{a_{1}}{J^{2}} \left(\alpha \frac{\partial^{2} T_{1}}{\partial \xi^{2}} - 2\beta \frac{\partial^{2} T_{j}}{\partial \xi \partial \eta} + \gamma \frac{\partial^{2} T_{1}}{\partial \eta^{2}} \right) + \frac{a_{1}}{J^{3}} \left[\left(\alpha \frac{\partial^{2} x}{\partial \xi^{2}} \right) \left(z_{\xi} \frac{\partial T_{1}}{\partial \eta} - z_{\eta} \frac{\partial T_{1}}{\partial \xi} \right) + \alpha \frac{\partial^{2} z}{\partial \xi^{2}} - 2\beta \frac{\partial^{2} z}{\partial \xi \partial \eta} + \gamma \frac{\partial^{2} z}{\partial \eta^{2}} \left(-x_{\xi} \frac{\partial T_{1}}{\partial \eta} \right) \right] + \frac{1}{J} \left(x_{\xi} \frac{\partial T_{1}}{\partial \eta} \right) \frac{dz}{dt} \tag{10}$$

$$\frac{\partial T_{s}}{\partial t} = \frac{a_{s}}{J^{2}} \left(\alpha \frac{\partial^{2} T_{s}}{\partial \xi^{2}} - 2\beta \frac{\partial^{2} T_{s}}{\partial \xi \partial \eta} + \gamma \frac{\partial^{2} T_{s}}{\partial \eta^{2}} \right) + \frac{a_{s}}{J^{3}} \left[\left(\alpha \frac{\partial^{2} x}{\partial \xi^{2}} \right) \left(z_{\xi} \frac{\partial T_{s}}{\partial \eta} - z_{\eta} \frac{\partial T_{s}}{\partial \xi} \right) + \alpha \frac{\partial^{2} z}{\partial \xi^{2}} - 2\beta \frac{\partial^{2} z}{\partial \xi \partial \eta} + \gamma \frac{\partial^{2} z}{\partial \eta^{2}} \left(-x_{\xi} \frac{\partial T_{s}}{\partial \eta} \right) \right] + \frac{1}{J} \left(x_{\xi} \frac{\partial T_{s}}{\partial \eta} \right) \frac{dz}{dt}$$
(11)

$$\left\{\lambda_{s}\frac{1}{J}\left(x_{\xi}\frac{\partial T_{s}}{\partial\eta}\right) - \lambda_{l}\frac{1}{J}\left(x_{\xi}\frac{\partial T_{l}}{\partial\eta}\right)\right\}\left\{1 + \left(\frac{1}{J}\left[z_{\eta}\frac{\partial z_{mov}}{\partial\xi} - z_{\xi}\frac{\partial z_{mov}}{\partial\eta}\right]\right)^{2}\right\} = \rho_{s}L_{s}\frac{\partial z_{mov}}{\partial t}$$
(12)

References

- Beckett, G., MacKenzie, J. A., & Robertson, M. L. (2001). A moving mesh finite element method for the solution of two-dimensional Stefan problems. Journal of Computational Physics, 168(2), 500–518.
- Bonacina, C., Comini, G., Fasano, A., & Primicerio, M. (1973). Numerical solution of phase-change problems. International Journal of Heat and Mass Transfer, 16, 1825.
- Cao, W., Huang, W., & Russell, R. D. (1999). An r-adaptive finite element method based upon moving mesh PDEs. Journal of Computational Physics, 149, 221.
- Charn-Jung, K., & Kaviany, M. (1992). Numerical method for phase-change problems with convection and diffusion. *International Journal of Heat and Mass Transfer*, 35(2), 457–467.
- Chatterjee, A., & Prasad, V. (2000). A full three-dimensional adaptive finite volume scheme for transport and phase-change processes, Part I: Formulation and validation. *Numerical Heat Transfer; Part A: Applications*, 37(8), 801–821.
- Chellaiah, S., & Viskanta, R. (1988). Freezing of saturated and superheated liquid in porous media. *International Journal of Heat and Mass Transfer*, 31, 321–330.
 Cheung, F. B., Chawla, T. C., & Pedersen, D. R. (1984). The effect of heat generation and wall interaction on freezing and melting in a finite slab. *International Journal of Heat and Mass Transfer*, 27, 29.
- Cook, W. A. (1974). Body oriented coordinates for generating three-dimensional meshes. International Journal for Numerical Methods in Engineering, 8, 27–43.
- Coons, S.A. (1967). Surfaces for computer-aided design of space forms, Project MAC. Technical Rep. MAC-TR 44 MIT, MA, USA, Design Div., Dept. Mech. Eng., Available from: Clearing-house for Federal Scientific-Technical Information, National Bureau of Standards, Springfield, VA, USA.
- Crank, J., & Gupta, R. S. (1975). Isotherm migration method in two dimensions. International Journal of Heat and Mass Transfer, 18, 1101.

Crowley, A. B. (1978). Numerical solution of Stefan problems. International Journal of Heat and Mass Transfer, 21, 215.

- Duda, J. L., Malone, M. F., Notter, R. H., & Vrentas, J. S. (1975). Analysis of two-dimensional diffusion controlled moving boundary problems. International Journal of Heat and Mass Transfer, 18, 901.
- Eriksson, L. E. (1982). Generation of boundary-conforming grid around wing-body configurations using transfinite interpolation. AIAA Journal, 20, 1313–1320.
- Ettouney, H. M., & Brown, R. A. (1983). Finite-element methods for steady solidification problems. Journal of Computational Physics, 49, 118.
- Frivik, P. E., & Comini, G. (1982). Seepage and heat flow in soil freezing. ASME Journal of Heat Transfer, 104, 323-328.
- Gong, Z. X., & Mujumdar, A. S. (1998). Flow and heat transfer in convection-dominated melting in a rectangular cavity heated from below. *International Journal of Heat and Mass Transfer*, 41(17), 2573–2580.
- Gordon, W. J., & Hall, C. A. (1973). Construction of curvilinear coordinate systems and applications to mesh generation. *International Journal for Numerical Methods in Engineering*, 7, 461–477.
- Hasan, M., Mujumdar, A. S., & Weber, M. E. (1991). Cyclic melting and freezing. Chemical Engineering Science, 46(7), 1573–1587.
- Hashemi, H. T., & Sliepcevich, C. M. (1973). Effect of seepage stream and artificial soil freezing. ASCE Journal of Mechanics Foundry Division, 99, 267–289.
- Hsu, C. F., Sparrow, E. M., & Patankar, S. V. (1981). Numerical solution of moving boundary problems by boundary immobilization and a control-volume-based finite difference scheme. *International Journal of Heat and Mass Transfer*, 24, 1335.
- John, D., & Anderson, Jr. (1995). Computational fluid dynamics. McGraw-Hill International Editions [Chapter 5].
- Khillarkar, D. B., Gong, Z. X., & Mujumdar, A. S. (2000). Melting of a phase change material in concentric horizontal annuli of arbitrary cross-section. *Applied Thermal Engineering*, 20(10), 893–912.
- Murray, W. D., & Landis, F. (1959). Numerical and machine solutions of transient heat conduction problem involving melting or freezing. ASME Journal of Heat Transfer, 81, 106–112.
- Rattanadecho, P. (2004a). Experimental and numerical study of solidification process in unsaturated granular packed bed. AIAA Journal of Thermophysics and Heat Transfer, 18(1), 87–93.
- Rattanadecho, P. (2004b). The theoretical and experimental investigation of Microwave thawing of frozen layer using microwave oven (effects of layered configurations and layered thickness). *International Journal of Heat and Mass Transfer*, 47(5), 937–945.
- Ratanadecho, P., Aoki, K., & Akahori, M. (2002). The characteristics of microwave melting of frozen packed bed using a rectangular wave guide. *IEEE Transaction of Microwave Theory and Techniques*, 50(6), 1487–1494.
- Saitoh, T. (1978). Numerical method for multi-dimensional freezing problems in arbitrary domains. Journal of Heat Transfer, 100, 294.
- Shamsundar, N., & Sparrow, E. M. (1976). Effect of density change on multi-dimensional conduction phase change. Journal of Heat Transfer, 97, 550.
- Sparrow, E. M., & Broadbent, J. A. (1983). Freezing in a vertical tube. ASME Journal of Heat Transfer, 105, 217–225.
- Sparrow, E. M., & Chuck, W. (1984). An implicit/explicit numerical solution scheme for phase-change problems. Numerical Heat Transfer, 7, 1.
- Sparrow, E. M., Ramadhyani, S., & Patankar, S. V. (1978). Effect of subcooling on cylindrical melting. ASME Journal of Heat Transfer, 100, 395.
- Weaver, J. A., & Viskanta, R. (1986). Freezing of liquid-saturated porous media. ASME Journal of Heat Transfer, 108, 654-659.